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ON STABLE COMPOSITE CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE ON THE SURFACE OF A FLUID OF FINITE DEPTH

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The problem of stable plane capillary-gravitational waves of finite amplitude on the surface of a perfect incompressible fluid stream of finite depth is considered. It is assumed that the waves are induced by pressure periodically distributed along the free surface, and that these, unlike induced waves, do not vanish when the pressure becomes constant, are transformed into free waves. Such waves are called composite; they exist similarly to free waves, for particular values of velocity of the stream.

The problem, which is rigorously stated, reduces to solving a system of four nonlinear equations for two functions and two constants. One of the equations is integral and the remaining are transcendental. Pressure on the surface is defined by an infinite trigonometric series whose coefficients are proportional to integral powers of some dimensionless small parameter; these powers are by two units greater than the numbers of coefficients.

The theorem of existence and uniqueness of solution is established, and the method of its proof is indicated. The derivation of solution in any approximation is presented in the form of series in powers of the indicated small parameter. Computation of the first three approximations is carried out to the end, and an approximate equation of the wave profile is presented.

Composite capillary-gravitational waves in the case of fluid of infinite depth were considered by the author in [1].

1. Statement of problem and derivation of basic equations. Let us consider a steady plane-parallel motion of a perfect incompressible heavy fluid of finite and constant depth h bounded from above by a free surface subjected to pressure $p_0 = p_0' + p_0(x)$, where $p_0' = \text{const}$ and $p_0(x)$ is a specified function of the horizontal coordinate x. We assume that the mean velocity c of the stream at the horizontal bottom is constant, is specified and directed from left to right. The term $p_0(x)$ indicates the presence of induced waves at any velocity c. In the absence of $p_0(x)$ free waves appear in the stream at certain particular values of c. Here it is assumed that pressure at the free surface is defined by the two terms. In this case the free surface in coordinates attached to the progressing wave moving at velocity c has the shape of a stationary periodic wave. We are seeking waves which do not vanish for $p_0(x) \equiv 0$ and for particular values of c are transformed into free waves. Such waves are called composite waves.

Let the unknown wave and the pressure $p_0(x)$ be symmetric about the vertical passing through the wave crest. We take this vertical line as the y-axis of an orthogonal system of coordinates xy. We take point O of intersection of the y-axis with the wave crest as the coordinate origin and direct the x- and y-axes to the right and upward, respectively.

We take the xy-plane of flow as the plane of the complex variable $z = x + iy$. We denote the velocity potential by φ, the stream function by ψ, the complex velocity potential by $w = \varphi + i\psi$, and the projections of the velocity vector $\mathbf{q}$ on the coordinate axes by U and V. We have

$$dw / dz = -U + iV, \qquad U = -\partial\varphi / \partial x, \qquad V = -\partial\varphi / \partial y$$

To derive the basic equations for this problem we, first, conformally map the region occupied by the wave and consisting of a vertical rectangle bounded from above by a wave-like curve on rectangle $0 \leqslant \varphi \leqslant c\lambda$, $0 \leqslant \psi \leqslant \psi_0$ in the w-plane (here $\psi = \psi_0 = ch$ is the stream flow rate per unit of time). This rectangle is then mapped on the interior of a circular ring whose center lies at the coordinate origin of the auxiliary complex plane $u = u_1 + iu_2$. The wave length λ is assumed to correspond to the periodicity of function $p_0(x)$. The last transformation is defined by formula

$$w = \frac{\lambda c}{2\pi i} \ln u \tag{1.1}$$

With this transformation the segment $0 \leqslant \varphi \leqslant c\lambda$ which corresponds to the free surface becomes the circumference of the outer circle of radius unity, and the segment corresponding to the bottom becomes the circumference of the inner circle of radius $r_0 = \exp(-2\pi\psi_0 / \varphi_0) = \exp(-2\pi h / \lambda)$ which is smaller than unity. The ring is slit along segment $(r_0, 1)$. It is assumed that h and λ, and consequently, also, r_0 are specified. The image of this ring of the u-plane onto the region of one wave in the z-plane is determined by formula

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{e^{i\omega(u)}}{u}, \qquad \omega(u) = \Phi + i\tau \tag{1.2}$$

From (1.1) and (1.2) we obtain $dw / dz = -ce^{\tau - i\Phi}$. This implies that throughout the stream, function Φ is equal to the angle between the velocity vector $\mathbf{q}$ and the x-axis, and

$$q = |\mathbf{q}| = ce^{\tau} \tag{1.3}$$

Since function $\omega(u)$ is holomorphic, it can be represented inside the considered ring of the u-plane by a Laurent series. It can be shown that the coefficients of that series must be real in virtue of the symmetry of the wave and pressure $p_0(x)$. It is also possible to satisfy the boundary condition at the bottom.

For $u = e^{i\theta}$ (θ is the angle between the radius vector and the u_1-axis) (1.2) yields a differential relation which, after separation of real and imaginary parts and integration, yields for the wave shape the parametric equation

$$x = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta \tag{1.4}$$

$$\tau(\eta) = \tau(1, \eta), \quad \Phi(\eta) = \Phi(1, \eta)$$

It follows from (1.4) that solution of the problem requires the determination of $\tau(\theta)$ in addition to $\Phi(\theta)$. The expansion of function $\omega(u)$ shows that these functions can be represented by the following trigonometric series:

$$-\tau(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \quad \Phi(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta \tag{1.5}$$

Expansions (1.5) satisfy the condition of wave symmetry about the vertical passing through its crest. From the Laurent expansion of function $\omega(u)$, which satisfies the condition of flow at the bottom, we obtain

$$A_n = \frac{\nu_n''}{n} B_n \quad (n = 1, 2, \ldots) \tag{1.6}$$

where ν_n'' are determined by formulas (1.18). Thus, if B_n are known, it is possible to determine by (1.6) all A_n, except A_0.

To determine the boundary condition at the surface we use the Bernoulli integral

$$p / \rho = C - gy - {}^1/_2 q^2 \tag{1.7}$$

where C is a constant, g is the acceleration of gravity, and ρ is the density. At the free surface the pressure difference is balanced by the normal component of surface tension force. For these forces by the Laplace law we have

$$p - p_0 = \pm \mu / R \tag{1.8}$$

where p is the pressure from the fluid side, $p_0 = p_0' + p_0(x)$ is the pressure from the free surface side, μ is the capillary constant, and R is the radius of curvature at points of the surface. Expressing curvature in terms of $d\Phi / d\theta$ from (1.8) we obtain

$$p - p_0 = \frac{2\pi\mu}{\lambda c} q \frac{d\Phi}{d\theta} \tag{1.9}$$

Substituting p defined by (1.9) into (1.7) and allowing for (1.3), we obtain

$$\frac{d\Phi}{d\theta} = \nu\left[\delta e^{-\tau} - e^{\tau} - \frac{2\pi}{\lambda} \varkappa y e^{-\tau} - p_0^*(x) e^{-\tau}\right] \tag{1.10}$$

$$\nu = \frac{\lambda c^2 \rho}{4\pi\mu}, \quad \delta = \frac{2(C\rho - p_0')}{\rho c^2}, \quad \varkappa = \frac{g\lambda}{\pi c^2}, \quad p_0^*(x) = \frac{2p_0(x)}{\rho c^2} \tag{1.11}$$

where x and y are determined by formulas (1.4) in terms of θ. We separate in the right-hand part of (1.10) terms that are linear with respect to Φ and τ taking into account the formula for y, and obtain

$$\frac{d\Phi}{d\theta} = \nu\left\{\delta - 1 + (\delta + 1)\tau + \varkappa \int_0^\theta \Phi(\eta)\, d\eta - S(\theta)(1 - \tau) + F[\tau, \Phi, S, \delta]\right\} \tag{1.12}$$

$$F[\tau, \Phi, S, \delta] = \delta(e^{-\tau} - 1 + \tau) - (e^{\tau} - 1 - \tau) + \varkappa e^{-\tau} \int_0^\theta [e^{-\tau(\eta)} \sin \Phi(\eta) - \Phi(\eta)]\, d\eta - \varkappa \int_0^\theta \Phi(\eta)\, d\eta + \varkappa e^{-\tau} \int_0^\theta \Phi(\eta)\, d\eta - S(\theta)(e^{-\tau} - 1 + \tau)$$

It is assumed here that

$$p_0^*(x) = \sum_{n=1}^{\infty} \varepsilon^{n+2} a_n \cos \frac{2\pi n}{\lambda} x, \quad S(\theta) = p_0^*[x(\theta)] \tag{1.13}$$

where ε is a small positive dimensionless parameter and d_n are specified real numbers, is correct to within the constant included in p_0' , and the series $\Sigma \varepsilon^n d_n$ converges in circle $\varepsilon_0 > 0$. To obtain $S(\theta)$ it is necessary to substitute into (1.13) values of $x(\theta) / \lambda$ defined by the equation

$$\frac{x(\theta)}{\lambda} = -\frac{1}{2\pi} \int_0^{\theta} e^{-\tau(\eta)} \cos \Phi(\eta)\, d\eta \tag{1.14}$$

which follows from (1.4).

Let us determine more accurately the formulas for parameters. In the case of free waves $S(\theta) \equiv 0$ and, as can be shown, it is necessary to set $c^2 = c_*^2(1 - \varepsilon^2)$, where c_*^2 is defined for a free linear capillary-gravitational wave of length λ by the following formula [2]:

$$c_*^2 = \left(\frac{2\pi\mu}{\lambda\rho} + \frac{g\lambda}{2\pi}\right) \operatorname{th}\left(2\pi \frac{h}{\lambda}\right) \tag{1.15}$$

Taking into account the definition (1.11) for c, we obtain

$$\nu = \frac{\lambda\rho}{4\pi\mu} c_*^2 (1 - \varepsilon^2) = \nu^{(0)}(1 - \varepsilon^2), \quad \nu^{(0)} = \frac{\lambda\rho c_*^2}{4\pi\mu} \tag{1.16}$$

$$\varkappa = \frac{g\lambda}{\pi c_*^2 (1 - \varepsilon^2)} = \varkappa_0 \left(1 + \sum_{n=1}^{\infty} \varepsilon^{2n}\right), \quad \varkappa_0 = \frac{g\lambda}{\pi c_*^2}$$

The substitution of these expressions into (1.12) yields

$$\frac{d\Phi}{d\theta} = \nu^{(0)} \Big\{\delta - 1 + (\delta + 1)\tau + \varkappa_0 \int_0^{\theta} \Phi(\eta)\, d\eta + \tag{1.17}$$

$$\varkappa_0 \sum_{n=1}^{\infty} \varepsilon^{2n} \int_0^{\theta} \Phi(\eta)\, d\eta - S(\theta)(1 - \tau) + F[\tau, \Phi, S, \delta]\Big\} - \nu^{(0)}\varepsilon^2 \{\ldots\}$$

where the expression omitted in the second set of braces is the same as in the first set.

Formula (1.17) determines the relation between functions $\Phi(\theta)$ and $\tau(\theta)$ at the circumference $|u| = 1$ of the ring plane u. It is shown in the theory of analytic functions that along that particular circumference the following formulas which follow from Villat's formulas [3] for a ring and generalize Dini's relationships for a circle:

$$-\tau(\theta) - A_0 = \int_0^{2\pi} K(\eta, \theta) \frac{d\Phi}{d\eta}\, d\eta, \quad K(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n'} \tag{1.18}$$

$$\Phi(\theta) = \int_0^{2\pi} K_0(\eta, \theta) \frac{d\tau}{d\eta}\, d\eta, \quad K_0(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{\nu_n''}$$

$$\nu_n' = n \frac{1 - r_0^{2n}}{1 + r_0^{2n}} = n \operatorname{th}\left(2\pi n \frac{h}{\lambda}\right), \quad \nu_n'' = \frac{1 + r_0^{2n}}{1 - r_0^{2n}} =$$

$$n \operatorname{cth}\left(2\pi n \frac{h}{\lambda}\right), \quad \nu_n' \nu_n'' = n^2$$

We transform the terms linear with respect to τ, Φ and ε, which appear in braces, using formulas (1.18) and integrating by parts. We then combine in the first braces the terms (with coefficients 2 and $-\varkappa_0$) with the same integrand $d\Phi / d\eta$ and different kernels

$$K(\theta, \eta), \quad K_2(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{n^2}$$

The constants $\nu^{(0)}$ and $\varkappa_0$ that appear in Eq. (1.17) are assumed to be specified, and δ is determined by the condition of periodicity of $\Phi(\theta + 2\pi) = \Phi(\theta)$. Since the right-hand part of Eq. (1.17) contains ε, its solutions, as well as δ depend on ε. Let us set

$$\delta = \delta_0 + \delta'(\varepsilon) \tag{1.19}$$

The condition of periodicity for $\varepsilon \to 0$ implies that $\delta_0 = 1$, since then the quantity $\delta'(\varepsilon)$ and also the solutions tend to zero. Equation (1.17) after all these transformations with allowance for (1.19) assumes the following final form:

$$\zeta(\theta) = \nu_1\left\{\int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\,d\eta + \delta'(\varepsilon) + (2 + \delta'(\varepsilon))A_0 + \right. \tag{1.20}$$

$$\left. \delta'(\varepsilon)\int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta + \varkappa_0\int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\,d\eta + \Psi(\theta, \varepsilon)\right\} -$$

$$\nu_1\varepsilon^2\left\{2\int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta - \varkappa_0\int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\,d\eta + \ldots\right\}$$

$$\zeta(\theta, \varepsilon) = \frac{d\Phi}{d\theta}, \quad \Psi(\theta, \varepsilon) = \varkappa_0\sum_{n=1}^{\infty}\varepsilon^{2n}\int_0^{\theta}\Phi(\eta)\,d\eta -$$

$$S(\theta)\left[1 + A_0 + \int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta\right] + F[\tau, \Phi, S, 1 + \delta'(\varepsilon)]$$

$$K^*(\eta, \theta) = \sum_{n=1}^{\infty}\frac{\varphi_n(\eta)\varphi_n(\theta)}{\nu_n}, \quad \nu_n = \frac{n^2}{2\nu_n'' - \varkappa_0}, \quad \varphi_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}$$

where ν_n are eigenvalues, $\varphi_n(\theta)$ are eigenfunctions of kernel $K^*(\eta, \theta)$, and the dots in the second set of braces replace the last six terms identical to those in the first set. Furthermore it is assumed that $\nu^{(0)} = \nu_1$, and that parameter $\varkappa_0$ is such that the eigenvalue ν_1 is simple and positive [2].

Note that for $\nu^{(0)} = \nu_1$ we obtain the required formula for c_*^2, since relationship (1.15) follows from formulas (1.20) and (1.16) for ν_n, and for $\nu^{(0)}$ and $\varkappa_0$, respectively.

The periodicity condition for function $\Phi(\theta)$ yields

$$\delta'(\varepsilon) = -\varkappa_0\int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\,d\eta - (2 + \delta'(\varepsilon))A_0 + \tag{1.21}$$

$$\varepsilon^2\left[\delta'(\varepsilon) + (2 + \delta'(\varepsilon))A_0 + \varkappa_0\int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\,d\eta\right] -$$

$$\frac{1 - \varepsilon^2}{2\pi}\int_0^{2\pi}\Psi(\theta, \varepsilon)\,d\theta$$

As previously noted, it follows from (1.6) that having determined $d\Phi/d\theta$ and $\Phi(\theta)$, we obtain B_n and A_n for $n = 1, 2, \ldots$ It remains to determine A_0.

Setting in the right-hand part of the first of formulas (1.4) $\theta = 2\pi$, we must obtain in the left-hand part $-\lambda$, since then x decreases by λ. This yields for A_0 the following equation:

$$\exp(-A_0) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-\tau(\eta) - A_0]\cos\Phi(\eta)\,d\eta \qquad (1.22)$$

($-\tau(\eta) - A_0$ does not contain A_0 by virtue of (1.5)).

The problem thus reduces to the determination of functions $\zeta(\theta, \varepsilon) = d\Phi / d\theta$, $x(\theta, \varepsilon) / \lambda$, and constants $\delta = 1 + \delta'(\varepsilon)$ and $A_0(\varepsilon)$ from the system of nonlinear equations (1.14), (1.20), (1.21) and (1.22). Parameter $\tau(\theta, \varepsilon)$ is obtained from (1.18), and

$$\Phi(\theta, \varepsilon) = \int_0^{\theta} \zeta(\eta, \varepsilon)\,d\eta \qquad (1.23)$$

By eliminating $x(\theta, \varepsilon)$ and $A_0(\varepsilon)$ from Eqs. (1.20) and (1.21) with the use of (1.14) and (1.22), and taking $\tau(\theta, \varepsilon)$ as defined in (1.18) and $\Phi(\theta, \varepsilon)$ in (1.23), the system of equations reduces to two: (1.20) and (1.21). Equation (1.20) is a nonlinear integral equation in $\zeta(\theta, \varepsilon)$ with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)} = \nu_1$, and is transcendental with respect to $\delta'(\varepsilon)$. Equation (1.21) is nonlinear and transcendental with respect to $\delta'(\varepsilon)$ with a linear functional relative to the unknown function. It is, however, more convenient not to carry out this transformation and to consider the system of four equations. In that case the only nonlinear integral equation is (1.20) in $\zeta(\theta, \varepsilon)$, while the remaining, including (1.20) are to be considered as nonlinear transcendental equations with respect to $x(\theta, \varepsilon) / \lambda$ and constants $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ with linear operators and functionals relative to the unknown functions.

2. Solution of basic equations of the problem. Solution of the system of Eqs. (1.14), (1.20), (1.21) and (1.22) is sought in the form of series in powers of parameter ε. For each coefficient of the expansion of function $\zeta(\theta, \varepsilon)$ we obtain a Fredholm linear integral equation of the second kind with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)} = \nu_1$ as the first eigenvalue of the latter. For the first coefficient of this expansion we obtain a homogeneous integral equation which is solved by the first Fredholm theorem. For all subsequent approximations we have nonhomogeneous equations which are solved by the third Fredholm theorem. The solutions of each of such equations is expressed in the form of the sum of solutions of the first homogeneous equation with an indeterminate coefficient C_{1n} (for the n-th approximation) and of the particular solution of the nonhomogeneous equation. Coefficient C_{1n} is determined by the condition of solvability of the equation for the $(n + 2)$-nd approximation. Thus each of the coefficients C_{11}, C_{12} and C_{13} is determined by the condition of solvability of equations for the third, fourth and fifth approximation.

For the coefficients of expansions of remaining unknown quantities we obtain a system of linear algebraic equations. This system which is always solvable yields for the coefficients of a particular approximation explicit expressions in terms of quantities derived in preceding approximations.

2.1. Determination of the first three approximations. We present third approximation formulas for $\zeta(\theta, \varepsilon)$, $x(\theta, \varepsilon) / \lambda$, $\delta'(\varepsilon)$ and $A_0(\varepsilon)$

$$\zeta(\theta, \varepsilon) = \varepsilon C_{11}\cos\theta + \varepsilon^2 C_{22}\cos 2\theta + \varepsilon^3 (C_{13}\cos\theta + C_{33}\cos 3\theta) \quad (2.1)$$

$$x(\theta, \varepsilon)/\lambda = -\frac{\varepsilon}{2\pi\nu_1'} C_{11}\sin\theta - \frac{\varepsilon^2}{16\pi}\left(\frac{1+\nu_1'^2}{\nu_1'^2} C_{11}^2 + \frac{4}{\nu_2'} C_{22}\right) \times$$
$$\sin 2\theta + \varepsilon^3 x_3(\theta)$$

$$\delta'(\varepsilon) = -\varepsilon\varkappa_0 C_{11} - \varepsilon^2\left(\frac{1}{4}\varkappa_0 C_{22} + 2A_{02} - \frac{1}{4}\frac{\varkappa_0}{\nu_1'} C_{11}^2\right) + \varepsilon^3\delta_3$$

$$A_0(\varepsilon) = \varepsilon^2 A_{02} = \varepsilon^2 \frac{1}{4}(1 - 1/\nu_1'^2) C_{11}^2$$

where

$$C_{12} = 0 \quad (\text{see } (2.5)), \quad C_{22} = -\frac{3}{4}\frac{\varkappa_0}{\nu_1'} C_{11}^2 \frac{\nu_1\nu_2}{(\nu_2 - \nu_1)} \quad (2.2)$$
$$C_{33} = C_{33}^* \frac{\nu_1\nu_3}{(\nu_3 - \nu_1)}$$

where C_{33}^* is a linear function of C_{11}^3 and $C_{11}C_{22}$; $x_3(\theta)$ is a linear function of $\sin\theta$ and $\sin 3\theta$ with coefficients that are linear with respect to C_{11}^3, $C_{11}C_{22}$ and C_{13} at $\sin\theta$ and, also, with respect to C_{11}^3, $C_{11}C_{22}$ and C_{33} at $\sin 3\theta$; δ_3 is a linear function of $C_{11}^3, C_{11}C_{22}, C_{11}, C_{13}$ and C_{33}; $A_{01} = A_{03} = 0$; coefficient C_{13} was not computed, since the fifth approximation required for its determination was not calculated; C_{11} is obtained from the equation (2.3)

$$\beta C_{11}^3 - \frac{2}{\nu_1'} C_{11} - d_1 = 0, \quad \beta = \frac{1}{4\nu_1'^3} + \frac{1-\nu_1'^2}{8\nu_1'^3}(4 + 3\varkappa_0\nu_1') + \frac{9\varkappa_0^2\nu_1\nu_2}{32\nu_1'^2(\nu_2 - \nu_1)}$$

Note that in the case of a free wave and $d_1 = 0$, (2.3) becomes the equation for C_{11}

2.2. Determination of further approximations. As previously stated, coefficient C_{12} is determined by the condition of solvability of the equation for $\zeta_4(\theta)$ which reduces to the equation

$$C_{12}C_{11}^2\left[\frac{1}{4\nu_1'^3}(10 - 7\nu_1'^2) - \frac{9}{8}\varkappa_0\frac{\nu_1'^2 - 1}{\nu_1'^2} + \varkappa_0^2\frac{\nu_1\nu_2(\nu_1'^2 + 2)}{\nu_1'(\nu_2 - \nu_1)}\right] = 0 \quad (2.4)$$

Hence

$$C_{12} = 0 \quad (2.5)$$

since $C_{11} \neq 0$ and, as can be readily shown, the expression in brackets is nonzero.

It can be shown by the method of mathematical induction that, as in the case of $n = 3$, we have $\zeta_n(\theta)$, $x_n(\theta)$, δ_n, and A_{0n} uniquely determined for any positive integral $n > 3$. The equation for C_{1n} is linear beginning with $n = 2$, and the coefficient at C_{1n} is the same as in (2.4).

3. Determination of the wave profile. The parametric form $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ of the wave profile equation us derived from formula (1.4) into which $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$ are to be substituted. We recall the functions $\tau(\theta, \varepsilon)$ and $\Phi(\theta, \varepsilon)$ are defined in terms of $\zeta(\theta, \varepsilon)$ by formulas (1.18) and (1.23). The elimination of θ from the parametric equations yields for the wave profile an equation of the form $y = y(x, \varepsilon)$.

Setting $k = 2\pi/\lambda$, we obtain for the wave profile the equation

$$y(x, \varepsilon) = \frac{1}{k}\Big\{\varepsilon C_{11}(\cos kx - 1) + \frac{\varepsilon^2}{4}\left(\frac{1}{\nu_1'} C_{11} - C_{22}\right)(1 - \cos 2kx) + \quad (3.1)$$
$$\frac{\varepsilon^3}{6}\Big[6C_{13} + \frac{9}{8}\left(1 - \frac{1}{\nu_1'^2}\right) C_{11}^3 + \frac{3}{2}\left(\frac{2}{\nu_1'} - \frac{1}{\nu_2'}\right) C_{11}C_{22} \times$$

$$(\cos kx - 1) + \frac{\varepsilon^3}{6}\left[\frac{2}{3}C_{33} + \frac{1}{8}\left(\frac{5}{\nu_1'^2} - \frac{7}{3}\right)C_{11}^3 - \left(\frac{1}{\nu_1'} + \frac{1}{2\nu_2'}\right)\times\right.$$
$$\left.C_{11}C_{22}(\cos 3kx - 1)\right\}$$

accurate to within third order terms, where coefficients C_{ij} are determined by formulas (2.2), (2.3) and (2.5).

Note. Since in accordance with the statement of the problem in Sect. 1 the coordinate origin is located at the wave crest, hence for x close to zero y must be negative. The analysis of the principal term in (3.1) shows that this is so for $C_{11} > 0$. Because of this from Eq. (2.3) we find that we must have $d_1 > 0$. The above must be taken into consideration in the analysis of solutions of Eq. (2.3).

4. Existence and uniqueness of solution of the problem. The following theorem is established by the Liapunov-Schmidt methods and their development [4].

Theorem. The system of Eqs. (1.14), (1.20), (1.21) and (1.22) has the unique solution $\zeta(\theta, \varepsilon)$, $x(\theta, \varepsilon) / \lambda$, $A_0(\varepsilon)$ and $\delta'(\varepsilon)$ ($\delta'(\varepsilon) = \delta(\varepsilon) - 1$) which is small with respect to ε, continuous with respect to θ ($0 \leqslant \theta \leqslant 2\pi$), and is an analytic function of ε for $|\varepsilon| < \varepsilon_1 \leqslant \varepsilon_0$.

Proof of this theorem is similar to that presented in [5].

This theorem implies the absolute and uniform convergence of series for $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$. The convergence of series in powers of ε for the integrand functions in (1.4) follows from the general theorems of the analysis of the substitution of series into series. The convergence of series whose approximate sum is defined by formula (3.1) is determined by general theorems of analysis.

Note. When solving this problem, $p_0^*(x)$ was specified in the form (1.13), which made it possible to obtain the solution in the form of series in integral powers of parameter ε. If it is assumed that

$$p_0^*(x) = \sum_{n=1}^{\infty} \varepsilon^n d_n \cos\frac{2\pi n}{\lambda} x$$

it can be shown by the analysis of the bifurcation equation of the Liapunov-Schmidt method that the series would have to be in $\varepsilon^{1/2}$.

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